MTH 265 Lecture Notes, Week 3 Monday.

Remarks on the Integral Test; the Alternating Series Test; and Absolute Convergence.

Theorem 1. The Integral Test for Series (IT), *simplified.*

Let $f(x)$ be a function that is positive, continuous, and decreasing on $[N, \infty)$ for some $N \in \mathbb{Z}_{\geq 0}$.
Then, $\sum_{n=N}^{\infty} f(n)$ converges if and only if $L = \lim_{x\to\infty} \int f(x)\,dx$ converges.

↑ The change happens here. Instead of checking $\lim_{b\to\infty}\int_N^b f(x)\,dx$, we simply find the limit of the indefinite integral of $f(x)$. This is allowed since $F(N)$ will always be defined for $f(x)$ above.

Counterexample 1.1. The function $f(x)$ needs to be positive over $[N, \infty)$.

Let $f(x) = \sin(\pi x)$. Let $L = \int_0^{\infty} \sin(\pi x)\,dx = \lim_{t\to\infty}\left[-\frac{1}{\pi}\cos(\pi x)\right]_0^t = -\frac{1}{\pi}\lim_{t\to\infty}\left[\cos(t) - 1\right]$;

Observe that L diverges.

However, there is no $N \in \mathbb{Z}$ such that $f(x)$ is positive on $[N, \infty)$

since for any $k \in \mathbb{Z}$, $f(-\frac{1}{2}+2k) = \sin(\pi(-\frac{1}{2}+2k)) = \sin(-\frac{\pi}{2}+2\pi k) = \sin(-\frac{\pi}{2}) = -1$;

$\therefore$ The Integral Test cannot be applied.

Despite L diverging, $\sum_{n=0}^{\infty} \sin(\pi n) = \sum_{n=0}^{\infty}(0) = 0$ converges.

Counterexample 1.2. The function $f(x)$ needs to be decreasing over $[N, \infty)$.

Let $f(x) = \left(\sin^2(\pi x) + \frac{1}{x^2}\right)$; Then, $f(x)$ is positive and continuous for all $x \in \mathbb{R}$ with $x \neq 0$.

$$L = \lim_{x\to\infty}\int \sin^2(\pi x) + \frac{1}{x^2}\,dx = \ldots = \lim_{x\to\infty}\left[\frac{1}{2}\left(x - \frac{1}{2\pi}\sin(2\pi x)\right) - \frac{1}{x}\right] = \text{DNE}.$$

However, $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty}\left(\sin^2(\pi n) + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\left(0^2 + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges as a p-series with $p=2$.

Observe that $f(x)$ is not decreasing on $[N, \infty)$ for any $N \in \mathbb{Z}$.

$f'(x) = 2\pi\sin(\pi x)\cos(\pi x) + (-2)x^{-3}$;

For any $k \in \mathbb{Z}$, $f'(\frac{3}{4}+2k) = 2\pi\sin(\frac{3\pi}{4}+2\pi k)\cos(\frac{3\pi}{4}+2\pi k) = 2\pi\sin(\frac{3\pi}{4})\cos(\frac{3\pi}{4}) = -\pi$;

$\therefore$ The Integral Test cannot be applied.

Counterexample 1.3. The function $f(x)$ needs to be continuous over $[N, \infty)$.

Let $f(x) = (x - \frac{3}{2})^{-2}$; Observe that $f(x)$ has a vertical asymptote at $x = \frac{3}{2}$:

$$L_1 = \int_1^{1.5} f(x)\,dx = \lim_{b\to 1.5}\int_1^b (x-\tfrac{3}{2})^{-2}\,dx = \lim_{b\to 1.5}\left[-(x-\tfrac{3}{2})^{-1}\right]_1^b = \infty;\ L = \int_1^{\infty} f(x)\,dx = \infty.$$

However, $\sum_{n=1}^{\infty}\frac{1}{(n-\frac{3}{2})^2}$ converges by the Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$;

But: $f(x) = (x-\frac{3}{2})^{-2}$ is continuous, positive, and decreasing on $[2, \infty)$.

$\therefore$ The Integral Test can be used on $\sum_{n=2}^{\infty} f(n)$. Then, $L = \int_2^{\infty} f(x)\,dx = \lim_{b\to\infty}\left[-(x-\frac{3}{2})^{-1} + 2\right] = 2.$

Definition 2. Alternating Series.

A series $\sum_{n=n_0}^{\infty} a_n$ is an alternating series if for some sequence (b_n), $b_n \geq 0$ and $a_n = (-1)^{n-1} b_n$ for all $n \geq n_0$.

Remark. By reindexing, $\Sigma (-1)^n b_n$ is also an alternating series.

Theorem 3. The Alternating Series Test (AST).

Let $\sum_{n=n_0}^{\infty} (-1)^{n-1} b_n$ be an alternating series.

If all 3 conditions are satisfied:

① $b_n > 0$ for all $n \geq n_0$, i.e. all b_n are positive;

② $b_n \geq b_{n+1}$ for all $n \geq n_0$, i.e. (b_n) is decreasing; and

③ $\lim_{n\to\infty} b_n = 0$;

then $\sum_{n=n_0}^{\infty} (-1)^{n-1} b_n$ converges.

Example 3.1. $\Sigma_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is called the alternating harmonic series

Let $b_n = \frac{1}{n}$; Then, ① b_n is positive for all $n \geq 1$;

② For $n \geq 1$: $0 < n < n+1$ and $\frac{1}{n} > \frac{1}{n+1}$. $\therefore b_n > b_{n+1}$ for all $n \geq 1$.

③ $\lim_{n\to\infty}(\frac{1}{n}) = 0$;

By the Alternating Series Test, the alternating harmonic series converges.

Example 3.2. Let $b_n = ne^{-n}$; Show that $\Sigma_{n=1}^{\infty} (-1)^n b_n$ converges.

① For $n \geq 1$, b_n is positive.

② Let $f(x) = xe^{-x}$; Then, $f'(x) = xe^{-x}(-1) + e^{-x} = e^{-x}(-x+1)$; For $x \geq 2$, $f'(x)$ is negative.

$\therefore f(x)$ is decreasing on $[2, \infty)$; $\therefore (b_n) = (f(n))$ is a decreasing sequence.

③ $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{x}{e^x} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$; Then, $\lim\limits_{n\to\infty} b_n = 0$.

By the Alternating Series Test, $\Sigma_{n=1}^{\infty} (-1)^n ne^{-n}$ converges.

Example 3.3. Let $b_n = \frac{n!}{n^n}$ and consider $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$;

① For $n \geq 1$: b_n is positive.

② We want to show that $b_{n+1} \leq b_n$ for all $n \geq 1$.

For $n \geq 1$: $b_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)n!}{(n+1)(n+1)^n} = \frac{n!}{(n+1)^n} \leq \frac{n!}{n^n} = b_n$;

③ We can show $\lim_{n\to\infty} b_n = 0$ using the Squeeze Theorem.

For $n \geq 1$: $0 \leq b_n = \frac{n!}{n^n} = \frac{n(n-1)!}{n n^{n-1}} = \frac{(n-1)(n-2)!}{n \cdot n^{n-2}} \leq (1)\frac{(n-2)(n-3)!}{n \cdot n^{n-3}} \leq \cdots \leq (1)\frac{1}{n} = \frac{1}{n}$;

Therefore, $\lim\limits_{n\to\infty}(0) = 0 \leq \lim\limits_{n\to\infty} b_n \leq \lim\limits_{n\to\infty} \frac{1}{n} = 0$; $\therefore \lim\limits_{n\to\infty} b_n = 0$;

By the Alternating Series Test, $\Sigma_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

Example 3.4. Let $b_n = \frac{e^n}{n!}$ and consider $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^n}{n!}$;

① For $n \geq 1$: b_n is positive.

② WTS that $b_{n+1} \leq b_n$ for all $n \geq 3$. We'll look at a tail series instead.

$$b_{n+1} = \frac{e^{n+1}}{(n+1)!} = \frac{(e)(e^n)}{(n+1)n!} \leq (1)\frac{e^n}{n!} = b_n \quad \text{since } e \approx 2.718;$$

③ WTS $\lim_{n\to\infty} b_n = 0$ using the Squeeze Theorem.

Recall that $e < 3$. Then, for $n \geq 5$: $e^n \leq 3^n = 9(3^{n-2}) \leq 9(2)(3)^{n-2} \leq 9(n-2)!$;

Then, $0 < \frac{e^n}{n!} \leq \frac{9(n-2)!}{n!} = \frac{9(n-2)!}{(n)(n-1)(n-2)!} = \frac{9}{n(n-1)}$ for $n \geq 5$;

By the Squeeze Theorem: $\lim_{n\to\infty} 0 = 0 < \lim_{n\to\infty} \frac{e^n}{n!} = \lim_{n\to\infty} b_n < \lim_{n\to\infty} \frac{9}{n(n-1)} = 0$; $\therefore \lim_{n\to\infty} b_n = 0$;

By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1) \frac{e^n}{n!}$ converges.

Proposition 4. Let $\sum a_n$ be some series. If $\lim_{n\to\infty} |a_n| \neq 0$, then $\lim_{n\to\infty} a_n \neq 0$.

* This is useful when showing a series diverges using the Divergence Test.
Note that this does NOT involve the negation of the Alternating Series Test.

Example 4.1. Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \cos(\frac{1}{n})$ diverges.

Since $\lim_{n\to\infty} \cos(\frac{1}{n}) = \cos(0) = 1 \neq 0$, $\lim_{n\to\infty} (-1)^{n+1} \cos(\frac{1}{n}) \neq 0$;

By the Divergence Test, $\sum_{n=1}^{\infty} (-1)^{n+1} \cos(\frac{1}{n})$ diverges.

Example 4.2. Find all $p \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}$ converges.

Consider 2 cases: Case 1: Assume $p \leq 0$. Then, $\lim_{n\to\infty} \left(\frac{1}{n^p}\right) \neq 0$ and $\lim_{n\to\infty} (-1)^{n+1} \frac{1}{n^p} \neq 0$.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}$ diverges by the Divergence Test.

Case 2: Assume $p > 0$. Let $b_n = n^{-p}$;

① b_n is positive for all $n \geq 1$;

② Let $f(x) = x^{-p}$; Then, $f'(x) = (-p)x^{-p-1}$ and $f'(x)$ is negative on $(1, \infty)$.
Since $f(x)$ is decreasing on $[1, \infty)$, (b_n) must also be decreasing;

③ $\lim_{n\to\infty} \left(\frac{1}{n^p}\right) = 0$ since $p > 0$.

By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}$ converges.

Example 4.2. Let $b_n = \frac{n^n}{n!}$ and consider $\sum_{n=1}^{\infty}(-1)^{n-1} b_n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^n}{n!}$;

Similar to **Example 3.3**: $b_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)(n+1)^n}{(n+1)\,n!} = \frac{(n+1)^n}{n!} \geq \frac{n^n}{n!} = b_n > 0.$

Since (b_n) is increasing and positive, $\lim_{n\to\infty} b_n \neq 0$ and $\lim_{n\to\infty}(-1)^{n+1} b_n \neq 0$;

By the Divergence Test, $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ diverges.

Theorem 5. Alternating Series Estimation Theorem (ASET).

Let $S = \sum_{n=n_0}^{\infty}(-1)^{n-1} b_n$ be a series identified to be convergent by the Alternating Series Test.

Let $S_N = \sum_{n=n_0}^{N}(-1)^{n-1} b_n$ be the Nth partial sum of S. Then, $|R_N| = |S - S_N| \leq b_{n+1}$;

Remark. By reindexing, this also applies to $\sum_{n=n_0}^{\infty}(-1)^n b_n$;

Example 5.1. Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ correct to 3 decimal places.

① Let $b_n = \frac{1}{n!}$. Then, ① b_n is positive for all $n \geq 0$;

② for all $n \geq 0$: $0 < n! < (n+1)!$; Then, $b_n = \frac{1}{n!} > \frac{1}{(n+1)!} = b_{n+1}$;

③ Since $\lim_{n\to\infty}(n!) = \infty$, $\lim_{n\to\infty}\left(\frac{1}{n!}\right) = 0$;

By the Alternating Series Test, $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}$ converges.

② Using ASET, it suffices to find N such that $b_{N+1} < \frac{1}{2}(10^{-3})$;
Equivalently, find N such that $(N+1)! > 2(10^3) = 2000$;
By brute force: For $N=5$: $(N+1)! = 6! = 720$;
For $N=6$: $(N+1)! = 7! = 5040 > 2000$;
We need at least 6 terms.

③ Using a calculator: $\sum_{n=1}^{6}\frac{(-1)^n}{n!} = -\frac{91}{144} = -0.632$;

Example 5.2. Determine a bound on the error of $S_4 = \sum_{n=1}^{4}\frac{(-1)^n}{n^2} \approx -0.7986$ relative to $S = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$;

Let $b_n = \frac{1}{n^2}$; It can be shown that b_n is positive and decreasing for all $n \geq 1$ and $\lim_{n\to\infty}\frac{1}{n^2} = 0$;

Then, ASET applies and $|R_4| = |S - S_4| \leq b_{4+1} = \frac{1}{5^2} = 0.04$;

That is, the true value of S is in $[S_4 - 0.04, S_4 + 0.04] \approx [-0.8386, -0.7586]$;

Definition 6. A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
A series $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Proposition 7. If a series $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.
Furthermore, a series $\sum a_n$ can be only one of the three: (1) absolutely convergent;
(2) conditionally convergent; or
(3) divergent;

EXAMPLE 7.1. The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is conditionally convergent.

EXAMPLE 7.2. From Example 4.2: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}$ is $\begin{cases} \text{divergent} & \text{if } p \le 0 \\ \text{conditionally convergent} & \text{if } 0 < p \le 1 \\ \text{absolutely convergent} & \text{if } p > 1 \end{cases}$;

Theorem 8. Let $\sum a_n$ be some series and let $\sum b_n$ be some rearrangement or regrouping of $\sum a_n$.
If $\sum a_n$ is absolutely convergent, then $\sum b_n$ is absolutely convergent with $\sum b_n = \sum a_n$;

EXAMPLE 8.1. Determine if $S = \sum_{n=1}^{\infty}\frac{1}{n(n+3)}$ converges and if it does, find its sum.

① $\sum_{n=1}^{\infty}\frac{1}{n(n+3)}$ converges by the Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$;

② Since $\left|\frac{1}{(n)(n+3)}\right| = \frac{1}{n(n+3)}$ for $n \ge 1$; $\sum_{n=1}^{\infty}\frac{1}{n(n+3)}$ is absolutely convergent.
Therefore, we can regroup the terms and the sum will not change.

③ Let $a_n = \frac{1}{n(n+3)}$;

By partial fraction decomposition: $\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} = \frac{1}{3}\left(\frac{1}{n}\right) - \frac{1}{3}\left(\frac{1}{n+3}\right)$;

$1 = A(n+3) + B(n)$; If $n=-3$: $B = -\frac{1}{3}$; If $n=0$: $A = \frac{1}{3}$;

Let $f(n) = \frac{1}{3}\left(\frac{1}{n}\right)$; Then, $a_n = f(n) - f(n+3)$;

Observe that by grouping the $(6n+1)^{th}$ to the $(6n+6)^{th}$ terms, we get:

$$\sum_{k=6n+1}^{6n+6} a_n = \begin{aligned}[t] & f(6n+1) - \cancel{f(6n+4)} + f(6n+2) - \cancel{f(6n+5)} + f(6n+3) - \cancel{f(6n+6)} \\ & + \cancel{f(6n+4)} - f(6n+7) + \cancel{f(6n+5)} - f(6n+8) + \cancel{f(6n+6)} - f(6n+9) \end{aligned}$$
$$= \big(f(6n+1) + f(6n+2) + f(6n+3)\big) - \big(f(6n+7) + f(6n+8) + f(6n+9)\big);$$

Let $g(n) = f(6n+1) + f(6n+2) + f(6n+3)$ and let $b_n = a_{6n+1} + \cdots + a_{6n+6} = g(n) - g(n+1)$;

Then, $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} g(n) - g(n+1) = \lim_{n\to\infty} g(0) - g(n+1) = g(0) + 0 = f(1) + f(2) + f(3)$

↑ This is a telescoping sum with partial sum $P_n = g(0) - g(n+1)$;

$$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \boxed{\frac{11}{18}}$$